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The complex-type Padovan$-p$ sequences

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Abstract. In this paper, we define the complex-type Padovan-p sequence and then give the relationships between the Padovan-p numbers and the complex-type Padovan-p numbers. Also, we provide a new Binet formula and a new combinatorial representation of the complex-type Padovan-p numbers by the aid of the nth power of the generating matrix of the complex-type Padovan-p sequence. In addition, we derive various properties of the complex-type Padovan-p numbers such as the permanental, determinantal and exponential representations and the finite sums by matrix methods.

1. Introduction and Preliminaries

The Padovan p-numbers $\{Pap\,(n)\}$ for any given p $(p = 2, 3, 4, \ldots)$ is defined [4] by the following homogeneous linear recurrence relation:

$$Pap\,(n + p + 2) = Pap\,(n + p) + Pap\,(n)\,, \tag{1}$$

for $n \geq 1$, with initial conditions $Pap\,(1) = Pap\,(2) = \cdots = Pap\,(p) = 0$, $Pap\,(p + 1) = 1$ and $Pap\,(p + 2) = 0$. When $p = 1$ in (1), the Padovan p-numbers $\{Pap\,(n)\}$ is reduced to the usual Padovan sequence $\{P\,(n)\}$.

The complex Fibonacci sequence $\{F_n^*\}$ is defined [7] by a two-order recurrence equation:

$$F_n^* = F_n + iF_{n+1},$$

for $n \geq 0$, where $\sqrt{-1} = i$ and F_n is the n^{th} Fibonacci number (cf. [1, 8]).

Kalman [10] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding k terms

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [10], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix

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method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

In the literature, many interesting properties and applications of the recurrence sequences relevant to this paper have been studied by many authors, for example [9,11–13,16–22]. In particular, in [5] and [6], the authors defined the new sequences using the quaternions and complex numbers, and then they gave miscellaneous properties and many applications of the sequences defined. In this work, we define the complex-type Padovan-p sequence. Also, give the relationships between the Padovan-p numbers and the complex-type Padovan-p numbers, and then we obtain generating a matrix of the complex-type Padovan-p sequence. Furthermore, we produce the Binet formula for this defined sequence. Finally, we give various properties of the complex-type Padovan-p numbers such as the combinatorial, permanental, determinantal and exponential representations, and the finite sums by matrix methods.

2. The Main Results

Now we define a new sequence that we call the complex-type Padovan-p sequence $\left\{Pa_p^{(i)}(n)\right\}$ as follows:

$$Pa_p^{(i)}(n+p+2) = i^2 \cdot Pa_p^{(i)}(n+p) + i^{p+2} \cdot Pa_p^{(i)}(n), \tag{2}$$

for any given p $(p = 3, 5, 7, \ldots)$ and $n \geq 1$, where $Pa_p^{(i)}(1) = \cdots = Pa_p^{(i)}(p) = 0$, $Pa_p^{(i)}(p+1) = 1$, $Pa_p^{(i)}(p+2) = 0$ and $\sqrt{-1} = i$. From the relations in the definitions of the complex-type Padovan-p numbers and the Padovan-p numbers, we derive the following relations:

$$Pa_p^{(i)}(n) = \begin{cases} i^{p+1} \cdot Pap(n), & \text{for } n \equiv 0 \pmod 4, \\ i^{p+2} \cdot Pap(n), & \text{for } n \equiv 1 \pmod 4, \\ i^{p+3} \cdot Pap(n), & \text{for } n \equiv 2 \pmod 4, \\ i^{p} \cdot Pap(n), & \text{for } n \equiv 3 \pmod 4. \end{cases}$$

From the equation (2), we can write the following companion matrix:

$$D_p = \left[d_{jk}^{(p)}\right]_{(p+2)\times(p+2)} = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & i^{p+2} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

The matrix D_p is said to be the complex-type Padovan-p matrix. Then we can write the following matrix relation:

$$\begin{bmatrix} Pa_p^{(i)}(n+p+2) \\ Pa_p^{(i)}(n+p+1) \\ \vdots \\ Pa_p^{(i)}(n+2) \\ Pa_p^{(i)}(n+1) \end{bmatrix} = D_p \cdot \begin{bmatrix} Pa_p^{(i)}(n+p+1) \\ Pa_p^{(i)}(n+p) \\ \vdots \\ Pa_p^{(i)}(n+1) \\ Pa_p^{(i)}(n) \end{bmatrix}.$$

It can be readily established by mathematical induction that for $n \geq p+1$,

$$(D_p)^n = \left[\begin{matrix} Pa_p^{(i)}(n+p+1) & Pa_p^{(i)}(n+p+2) & i^{p+2}\cdot Pa_p^{(i)}(n+1) \\ Pa_p^{(i)}(n+p) & Pa_p^{(i)}(n+p+1) & i^{p+2}\cdot Pa_p^{(i)}(n) \\ Pa_p^{(i)}(n+p-1) & Pa_p^{(i)}(n+p) & i^{p+2}\cdot Pa_p^{(i)}(n-1) \\ \vdots & \vdots & \vdots \\ Pa_p^{(i)}(n+1) & Pa_p^{(i)}(n+2) & i^{p+2}\cdot Pa_p^{(i)}(n-p+1) \\ Pa_p^{(i)}(n) & Pa_p^{(i)}(n+1) & i^{p+2}\cdot Pa_p^{(i)}(n-p) \end{matrix}\right.$$

$$\left.\begin{matrix} i^{p+2}\cdot Pa_p^{(i)}(n+2) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p) \\ i^{p+2}\cdot Pa_p^{(i)}(n+1) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p-1) \\ i^{p+2}\cdot Pa_p^{(i)}(n) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p-2) \\ \vdots & \ddots & \vdots \\ i^{p+2}\cdot Pa_p^{(i)}(n-p+2) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n) \\ i^{p+2}\cdot Pa_p^{(i)}(n-p+1) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n-1) \end{matrix}\right],$$

from which it is clear that $\det D_p = i^{p+2}$. For more information on the companion matrices, see [14, 15].

Using the $(D_p)^n$ matrix, we determine the following relationships between complex-type Padovan-p numbers and the Padovan-p sequence for $n \geq p+1$

such that every odd integer where $p \geq 3$:

$$(D_p)^n = \begin{bmatrix} i^{n+4}\cdot Pap\,(n+p+1) & i^{n+5}\cdot Pap\,(n+p+2) & i^{n+2}\cdot Pap\,(n+1) & i^{n+3}\cdot Pap\,(n+2) & \cdots & i^{n+p+1}\cdot Pap\,(n+p) \\ i^{n+3}\cdot Pap\,(n+p) & i^{n+4}\cdot Pap\,(n+p+1) & i^{n+1}\cdot Pap\,(n) & i^{n+2}\cdot Pap\,(n+1) & \cdots & i^{n+p}\cdot Pap\,(n+p-1) \\ i^{n+2}\cdot Pap\,(n+p-1) & i^{n+3}\cdot Pap\,(n+p) & i^{n}\cdot Pap\,(n-1) & i^{n+1}\cdot Pap\,(n) & \cdots & i^{n+p-1}\cdot Pap\,(n+p-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ i^{n-p+4}\cdot Pap\,(n+1) & i^{n-p+5}\cdot Pap\,(n+2) & i^{n-p+2}\cdot Pap\,(n-p+1) & i^{n-p+3}\cdot Pap\,(n-p+2) & \cdots & i^{n+1}\cdot Pap\,(n) \\ i^{n-p+3}\cdot Pap\,(n) & i^{n-p+4}\cdot Pap\,(n+1) & i^{n-p+1}\cdot Pap\,(n-p) & i^{n-p+2}\cdot Pap\,(n-p+1) & \cdots & i^{n}\cdot Pap\,(n-1) \end{bmatrix}.$$

Now we concentrate on finding the Binet formulas for the complex-type Padovan-p numbers.

Lemma 1. *Let p be a positive odd integer such that $p \geq 3$. The characteristic equation of the complex-type Padovan-p numbers $x^{p+2} + x^p - i^{p+2} = 0$ does not have multiple roots.*

Proof. Let $f(x) = x^{p+2} + x^p - i^{p+2}$. It is clear that $f(0) \neq 0$ and $f(1) \neq 0$ for all $p \geq 3$. Let α be a multiple root of $f(x)$, then $\alpha \notin \{0, 1\}$. Since α is a multiple root,

$$f(\alpha) = \alpha^{p+2} + \alpha^p - i^{p+2} = 0$$

and

$$f'(\alpha) = (p+2)\,\alpha^{p+1} + p\alpha^{p-1} = 0,$$

hence

$$f'(\alpha) = \alpha^{p-1}\left((p+2)\,\alpha^2 + p\right) = 0.$$

Thus we obtain $\alpha = \pm\left(\frac{-p}{p+2}\right)^{\frac{1}{2}}$. Since p is a positive odd integer such that $p \geq 3$, $f(\alpha) \neq 0$, which is a contradiction. Thus, the equation $f(x) = 0$ does not have multiple roots. □

Let $f(x)$ be the characteristic polynomial of the matrix D_p. Then we have $f(x) = x^{p+2} + x^p - i^{p+2}$, which is a well-known fact from the companion matrices. If $\delta_1, \delta_2, \ldots, \delta_{p+2}$ are roots of the equation $x^{p+2} + x^p - i^{p+2} = 0$, then by Lemma 1, it is known that $\delta_1, \delta_2, \ldots, \delta_{p+2}$ are distinct. Define the $(p+2) \times (p+2)$ Vandermonde matrix V^{p+2} as follows:

$$V^{p+2} = \begin{bmatrix} (\delta_1)^{p+1} & (\delta_2)^{p+1} & \ldots & (\delta_{p+2})^{p+1} \\ (\delta_1)^{p} & (\delta_2)^{p} & \ldots & (\delta_{p+2})^{p} \\ \vdots & \vdots & & \vdots \\ \delta_1 & \delta_2 & \ldots & \delta_{p+2} \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Assume that $V_{j,k}^{p+2}$ is a $(p+2) \times (p+2)$ matrix obtained from the Vandermonde matrix V^{p+2} by replacing the j^{th} column of V^{p+2} by W_t^p, where W_t^p is a $(p+2) \times 1$ matrix as follows

$$W_t^p = \begin{bmatrix} (\delta_1)^{n+p+2-t} \\ (\delta_2)^{n+p+2-t} \\ \vdots \\ (\delta_{p+2})^{n+p+2-t} \end{bmatrix}.$$

Then we can give the generalized Binet formula for the complex-type Padovan-p numbers with the following theorem.

Theorem 1. *Let $n \geq p+1$ and let p be a positive odd integer such that $p \geq 3$, then*

$$d_{j,k}^{Pa,p,n} = \frac{\det V_{j,k}^{p+2}}{\det V^{p+2}},$$

where $(D_p)^n = \left[d_{j,k}^{Pa,p,n}\right]$.

Proof. Since the equation $x^{p+2} + x^p - i^{p+2} = 0$ does not have multiple roots for $p \geq 3$, when p is a positive odd integer, the eigenvalues of the complex-type Padovan-p matrix D_p are distinct. Then, it is clear that D_p is diagonalizable. Let $R_p = diag\,(\delta_1, \delta_2, \ldots, \delta_{p+2})$, then we may write $D_p V^{p+2} = V^{p+2} R_p$. Since the matrix V^{p+2} is invertible, we obtain the equation $\left(V^{p+2}\right)^{-1} D_p V^{p+2} = R_p$. Thus, D_p is similar to R_p; hence, $(D_p)^n V^{p+2} = V^{p+2} (R_p)^n$ for $n \geq p+1$. Therefore we have the following linear system of equations:

$$\begin{cases} d_{j,1}^{Pa,p,n} (\delta_1)^{p+1} + d_{j,2}^{Pa,p,n} (\delta_1)^p + \cdots + d_{j,p+2}^{Pa,p,n} = (\delta_1)^{n+p+2-t} \\ d_{j,1}^{Pa,p,n} (\delta_2)^{p+1} + d_{j,2}^{Pa,p,n} (\delta_2)^p + \cdots + d_{j,p+2}^{Pa,p,n} = (\delta_2)^{n+p+2-t} \\ \vdots \\ d_{j,1}^{Pa,p,n} (\delta_{p+2})^{p+1} + d_{j,2}^{Pa,p,n} (\delta_{p+2})^p + \cdots + d_{j,p+2}^{Pa,p,n} = (\delta_{p+2})^{n+p+2-t} . \end{cases}$$

Then we conclude that

$$d_{j,k}^{Pa,p,n} = \frac{\det V_{j,k}^{p+2}}{\det V^{p+2}},$$

for each $j, k = 1, 2, \ldots, p+2$. □

Thus by Theorem 1 and the matrix $(D_p)^n$, we have the following useful result for the complex-type Padovan-p numbers.

Corollary 1. *Let p be a positive odd integer such that $p \geq 3$ and $Pa_p^{(i)}(n)$ be the nth element of the complex-type Padovan-p number for $n \geq p+1$, then*

$$Pa_p^{(i)}(n) = \frac{\det V_{p+2,1}^{p+2}}{\det V^{p+2}}$$

and

$$Pa_p^{(i)}(n) = \frac{\det V_{2,3}^{p+2}}{i^{p+2} \cdot \det V^{p+2}} = \frac{\det V_{3,4}^{p+2}}{i^{p+2} \cdot \det V^{p+2}} = \cdots = \frac{\det V_{p+1,p+2}^{p+2}}{i^{p+2} \cdot \det V^{p+2}}.$$

Let $C(c_1, c_2, \ldots, c_v)$ be a $v \times v$ companion matrix as follows:

$$C(c_1, c_2, \ldots, c_v) = \begin{bmatrix} c_1 & c_2 & \cdots & c_{v-1} & c_v \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2 (Chen and Louck [3]). *The (i,j) entry $c_{i,j}^{(n)}(c_1, c_2, \ldots, c_v)$ in the matrix $C^n(c_1, c_2, \ldots, c_v)$ is given by the following formula:*
(3)

$$c_{i,j}^{(n)}(c_1, c_2, \ldots, c_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} c_1^{t_1} \cdots c_v^{t_v},$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (3) *are defined to be* 1 *if $n = i - j$.*

Here we investigate combinatorial representations for the complex-type Padovan-p numbers by the following corollary.

Corollary 2. (i) *For $n \geq p+1$,*

$$Pa_p^{(i)}(n) = \sum_{(t_1, t_2, \ldots, t_{p+2})} \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} (-1)^{t_2} \left(i^{p+2}\right)^{t_{p+2}},$$

where the summation is over nonnegative integers satisfying

$$t_1 + 2t_2 + \cdots + (p+2)\, t_{p+2} = n - p - 1.$$

(ii) *For $n \geq p+1$,*

$$\begin{aligned} Pa_p^{(i)}(n) &= \frac{1}{i^{p+2}} \sum_{(t_1, t_2, \ldots, t_{p+2})} \frac{t_3 + t_4 + \cdots + t_{p+2}}{t_1 + t_2 + \cdots + t_{p+2}} \times \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} (-1)^{t_2} \left(i^{p+2}\right)^{t_{p+2}} \\ &= \frac{1}{i^{p+2}} \sum_{(t_1, t_2, \ldots, t_{p+2})} \frac{t_4 + t_5 + \cdots + t_{p+2}}{t_1 + t_2 + \cdots + t_{p+2}} \times \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} (-1)^{t_2} \left(i^{p+2}\right)^{t_{p+2}} \end{aligned}$$

$$= \dots$$
$$= \frac{1}{i^{p+2}} \sum_{(t_1,t_2,\dots,t_{p+2})} \frac{t_{p+2}}{t_1+t_2+\cdots+t_{p+2}} \times \binom{t_1+t_2+\cdots+t_{p+2}}{t_1,t_2,\dots,t_{p+2}} (-1)^{t_2} \left(i^{p+2}\right)^{t_{p+2}},$$

where the summation is over nonnegative integers satisfying

$$t_1 + 2t_2 + \cdots + (p+2)\, t_{p+2} = n+1.$$

Proof. In Theorem 2, if we take $i = p+2$ and $j = 1$ for the case (i), and $i = \varepsilon - 1$ and $j = \varepsilon$ such that $3 \leq \varepsilon \leq p+2$ for the case (ii), then we can directly see the conclusions from $(D_p)^n$. □

Now we consider the permanental representations for the complex-type Padovan-p numbers.

Definition 1. A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k^{th} column (resp. row) if the k^{th} column (resp. row) contains exactly two non-zero entries.

Suppose that $x_1, x_2, \dots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [2], Brualdi and Gibson obtained that $per\,(M) = per\,(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the complex-type Padovan-p numbers and the permanents of certain matrices that are obtained by using the generating matrix of the Padovan-p numbers. Let p be a positive odd integer such that $p \geq 3$ and let $A^{(i)}_{p,m} = \left[a^{(p,i,m)}_{k,j}\right]$ be the $m \times m$ super-diagonal matrix, defined by

$$a^{(p,i,m)}_{k,j} = \begin{cases} i^{p+2}, & \text{if } k=\tau \text{ and } j=\tau+p+1 \text{ for } 1 \leq \tau \leq m-p-1, \\ 1, & \text{if } k=\tau+1 \text{ and } j=\tau \text{ for } 1 \leq \tau \leq m-1, \\ -1, & \text{if } k=\tau \text{ and } j=\tau+1 \text{ for } 1 \leq \tau \leq m-1, \\ 0, & \text{otherwise.} \end{cases} \quad \text{, for } m \geq p+2.$$

Then we have the following theorem.

Theorem 3. *For $m \geq p+2$ and $p \geq 3$,*

$$per A^{(i)}_{p,m} = Pa^{(i)}_p\,(m+p+1)\,.$$

Proof. Let us consider the matrix $A^{(i)}_{p,m}$ and let the equation be hold for $m \geq p+2$. We prove by induction on m. Then we show that the equation holds for $m+1$. If we expand the $A^{(i)}_{p,m}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$per A^{(i)}_{p,m+1} = -per A^{(i,k)}_{m-1} + i^{p+2} \cdot per A^{(i)}_{p,m-p-1}.$$

Since $perA^{(i)}_{p,m-1} = Pa^{(i)}_p(m+p)$ and $perA^{(i)}_{p,m-p-1} = Pa^{(i)}_p(m)$, it is clear that $perA^{(i)}_{p,m+1} = Pa^{(i)}_p(m+p+2)$. So the proof is complete. □

Let p be a positive odd integer such that $p \geq 3$ and let $B^{(i)}_{p,m} = \left[b^{(p,i,m)}_{k,j}\right]$ be the $m \times m$ super-diagonal matrix, defined by

$$b^{(p,i,m)}_{k,j} = \begin{cases} i^{p+2}, & \text{if } k=\tau \text{ and } j=\tau+p+1 \text{ for } 1 \leq \tau \leq m-p-1, \\ 1, & \text{if } k=\tau+1 \text{ and } j=\tau \text{ for } 1 \leq \tau \leq m-4 \text{ and} \\ & k=\tau \text{ and } j=\tau+1 \text{ for } m-2 \leq \tau \leq m-1, \\ -1, & \text{if } k=\tau \text{ and } j=\tau+1 \text{ for } 1 \leq \tau \leq m-3 \text{ and} \\ & k=\tau+1 \text{ and } j=\tau \text{ for } m-3 \leq \tau \leq m-1, \\ 0, & \text{otherwise,} \end{cases}$$

for $m \geq p+2$.

Then we have the following theorem.

Theorem 4. *For $m \geq p+2$ and $p \geq 3$,*

$$perB^{(i)}_{p,m} = -Pa^{(i)}_p(m+p+1).$$

Proof. Let us consider the matrix $B^{(i)}_{p,m}$ and let the equation be hold for $m \geq p+2$. We prove by induction on m. Then we show that the equation holds for $m+1$. If we expand the $B^{(i)}_{p,m}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perB^{(i)}_{p,m+1} = -perB^{(i)}_{m-1} + i^{p+2} \cdot perB^{(i)}_{p,m-p-1}.$$

Since $perB^{(i)}_{p,m-1} = -Pa^{(i)}_p(m+p)$ and $perB^{(i)}_{p,m-p-1} = -Pa^{(i)}_p(m)$, it is clear that $perB^{(i)}_{p,m+1} = -Pa^{(i)}_p(m+p+2)$. So the proof is complete. □

Assume next that $C^{(i)}_{p,m} = \left[c^{(p,i,m)}_{k,j}\right]$ be the $m \times m$ matrix, defined by

$$C^{(i)}_{p,m} = \begin{bmatrix} 1 & \cdots & \overset{\overset{(m-3)\,\text{th}}{\downarrow}}{1} & 0 & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & B^{(i)}_{m-1} & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}, \quad \text{for } m > p+2,$$

then we have the following results.

Theorem 5. *For $m > p+2$ and $p \geq 3$,*

$$perC^{(i)}_{p,m} = -\sum_{u=1}^{m+p} Pa^{(i)}_p(u).$$

Proof. If we extend $perC_{p,m}^{(i)}$ with respect to the first row, we write

$$perC_{p,m}^{(i)} = perC_{p,m-1}^{(i)} + perB_{p,m-1}^{(i)}.$$

Thus, by the results and an inductive argument, the proof is easily seen. □

A matrix M is called convertible if there is an $n \times n$ $(1,-1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the complex-type Padovan-p numbers and the determinants of certain matrices which are obtained by using the matrix $A_{p,m}^{(i)}$, $B_{p,m}^{(i)}$ and $C_{p,m}^{(i)}$. Let $m > p+2$ and let H be the $m \times m$ matrix, defined by

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{bmatrix}.$$

Corollary 3. *For $m > p+2$ and $p \geq 3$,*

$$\det\left(A_{p,m}^{(i)} \circ H\right) = Pa_p^{(i)}(m+p+1),$$

$$\det\left(B_{p,m}^{(i)} \circ H\right) = -Pa_p^{(i)}(m+p+1)$$

and

$$\det\left(C_{p,m}^{(i)} \circ H\right) = -\sum_{u=1}^{m+p} Pa_p^{(i)}(u).$$

Proof. Since $perA_m^{(i,k)} = \det\left(A_{p,m}^{(i)} \circ H\right)$, $perB_m^{(i,k)} = \det\left(B_{p,m}^{(i)} \circ H\right)$ and $perC_m^{(i,k)} = \det\left(C_{p,m}^{(i)} \circ H\right)$ for $m > p+2$, by Theorem 3, Theorem 4 and Theorem 5, we have the conclusion. □

It is easy to see that the generating function of the complex-type Padovan-p sequence $\left\{Pa_p^{(i)}(n)\right\}$ is as follows:

$$g(x) = \frac{x^{p+1}}{1 + x^2 - i^{p+2} \cdot x^{p+2}},$$

where p is a positive odd integer such that $p \geq 3$.

Now we are concerned about the exponential representation of the complex-type Padovan-p numbers by the aid of the generating function with the following theorem.

Theorem 6. *The complex-type Padovan-p sequence* $\left\{Pa_p^{(i)}(n)\right\}$ *have the following exponential representation:*

$$g(x) = x^{p+1}\exp\left(\sum_{u=1}^{\infty}\frac{(x)^u}{u}\left(-x+i^{p+2}\cdot x^{p+1}\right)^u\right),$$

where p is a positive odd integer such that $p \geq 3$.

Proof. Since

$$\ln g(x) = \ln x^{p+1} - \ln\left(1+x^2-i^{p+2}\cdot x^{p+2}\right)$$

and

$$\begin{aligned} -\ln\left(1+x^2-i^{p+2}\cdot x^{p+2}\right) &= -[-x\left(-x+i^{p+2}\cdot x^{p+1}\right) \\ &\quad -\frac{1}{2}x^2\left(-x+i^{p+2}\cdot x^{p+1}\right)^2-\cdots \\ &\quad -\frac{1}{u}x^u\left(-x+i^{p+2}\cdot x^{p+1}\right)^u-\cdots] \end{aligned}$$

it is clear that

$$g(x) = x^{p+1}\exp\left(\sum_{u=1}^{\infty}\frac{(x)^u}{u}\left(-x+i^{p+2}\cdot x^{p+1}\right)^u\right).$$

Thus we have the conclusion. □

Now we give the sums of the complex-type Padovan-p numbers. Let

$$S_n = \sum_{u=1}^{n} Pa_p^{(i)}(u),$$

for $n \geq p+1$ and p is a positive odd integer such that $p \geq 3$, and suppose that R_p is the $(p+3)\times(p+3)$ matrix such that

$$R_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & D_p & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

If we use induction on n, then we obtain

$$(R_p)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{n+p} & & & \\ S_{n+p-1} & & (D_p)^n & \\ \vdots & & & \\ S_{n-1} & & & \end{bmatrix}.$$

References

[1] G. Berzsenyi, *Sums of products of generalized Fibonacci numbers*, The Fibonacci Quarterly, 13 (4) (1975), 343–344.

[2] R. A. Brualdi, P. M. Gibson, *Convex polyhedra of doubly stochastic matrices I. Applications of permanent function*, The Journal of Combinatorial Theory, Series A, 22 (2) (1977), 194–230.

[3] W. Y. C. Chen, J. D. Louck, *The combinatorial power of the companion matrix*, Linear Algebra and its Applications, 232 (1996), 261–278.

[4] O. Deveci, E. Karaduman, *On the Padovan p-numbers*, Hacettepe Journal of Mathematics and Statistics, 46 (4) (2017), 579–592.

[5] O. Deveci, A. G. Shannon, *The complex-type k-Fibonacci sequences and their applications*, Communications in Algebra, 49 (3) (2021), 1352–1367.

[6] O. Deveci, A. G. Shannon, *The quaternion-Pell sequence*, Communications in Algebra, 46 (12) (2018), 5403–5409.

[7] A. F. Horadam, *A generalized Fibonacci sequence*, The American Mathematical Monthly, 68 (5) (1961), 455–459.

[8] A. F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, The American Mathematical Monthly, 70 (3) (1963), 289–291.

[9] A. F. Horadam, A. G. Shannon, *Ward's Staudt-Clausen problem*, Mathematica Scandinavica, 39 (2) (1976), 239–250.

[10] D. Kalman, *Generalized Fibonacci numbers by matrix methods*, The Fibonacci Quarterly, 20 (1) (1982), 73–76.

[11] E. Kilic, *The Binet formula, sums and representations of generalized Fibonacci p-numbers*, The European Journal of Combinatorics, 29 (3) (2008), 701–711.

[12] E. Kilic, D. Tasci, *On the generalized order-k Fibonacci and Lucas numbers*, The Rocky Mountain Journal of Mathematics, 36 (5) (2006), 1915–1926.

[13] E. Kilic, D. Tasci, *On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers*, The Rocky Mountain Journal of Mathematics, 37 (6) (2007), 1953–1969.

[14] P. Lancaster, M. Tismenetsky, *The theory of matrices: with applications*, Elsevier, 1985.

[15] R. Lidl, H. Niederreiter, *Introduction to finite fields and their applications*, Cambridge U.P., 1994.

[16] N. Y. Ozgur, *On the sequences related to Fibonacci and Lucas numbers*, Journal of the Korean Mathematical Society, 42 (1) (2005), 135–151.

[17] A. G. Shannon, *Explicit expressions for powers of arbitrary order linear recursive sequences*, The Fibonacci Quarterly, 12 (3) (1974), 281–287.

[18] A. G. Shannon, *Some properties of a fundamental recursive sequence of arbitrary order*, The Fibonacci Quarterly, 12 (4) (1974), 327–335.

[19] A. G. Shannon, *Ordered partitions and arbitrary order linear recurrence relations*, The Mathematics Student, 43 (3) (1976), 110–117.

[20] A. P. Stakhov, B. Rozin, *Theory of Binet formulas for Fibonacci and Lucas p-numbers*, Chaos, Solitons and Fractals, 27 (5) (2006), 1162–1177.

[21] A. P. Stakhov, B. Rozin, *The continuous functions for the Fibonacci and Lucas p-numbers*, Chaos, Solitons andFractals, 28 (4) (2006), 1014–1025.

[22] D. Tasci, M. C. Firengiz, *Incomplete Fibonacci and Lucas p-numbers*, Mathematical and Computer Modelling, 52 (9-10) (2010), 1763–1770.

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